Data 102 Fall 2026 **Midterm 1** **A. Strang**

NAME (FIRST LAST): _____ **SID:** _____

Room: ◯ Pimentel 1, ◯ Wheeler 20, ◯ Mulford 106, ◯ Mulford 107

Seat Number: _____

TIME: 110 minutes

Conditions: closed book/internet; no reference sheet; no calculator/computer

GRADING

- There are 9 questions and 60 points total.
- The exam will be graded for accuracy. Partial credit will be granted.
- Please mark all selection bubbles clearly. Fill in the bubble fully. Do not use x's or check marks.

YOUR WRITTEN WORK

- There is a space for your full name and SID number on each page. **You must fill this in.** It will ensure that we can identify your work during the scanning process.
- If you need scratch paper, please use the backs of the pages of the exam. Note: **only the front pages will be scanned and graded.**
- When working on short-answer problems, clearly indicate your answers, and justify your answers. We will give partial credit based on demonstrated knowledge. Make sure your work is organized and legible. We will not give partial credit for work we cannot follow.
- When working on short-answer problems from practice problems you must justify your answers as when solving the problems from scratch. Correct answers without justification will not receive full credit.
- Numerical answers may be left as mathematical expressions (need not be fully simplified) unless we explicitly ask for simplification. For example, $(6 \times 10 + 15 \times 3)/3$.

HONOR CODE

Exams are individual work. By signing on the line below you verify that all answers are your own and that you will not share the content of exams until answers are released. Unsigned exams may not be graded. **Please abide by UC Berkeley's Honor Code:**

"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

Your signature: _____

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Part 1: *Answer all parts of questions 1 - 5.*

Question 1 [6 points]. Consider a binary decision making problem. Let $R \in \{0, 1\}$ denote the possible states of reality and $D \in \{0, 1\}$ denote the decision returned by a decision making tool.

1. Define each error rate listed below as the conditional probability of an event:

   (a) The false positive rate equals $\Pr(D = 1|R = 0)$. [1 pt]

   (b) The false negative rate equals $\Pr(D = 0|R = 1)$. [1 pt]

   (c) The false discovery proportion equals $\Pr(R = 0|D = 1)$. [1 pt]

2. Select all true statements below:

   ☐ a decision procedure is sensitive if it has a false positive rate near zero

   ☐ a decision procedure is specific if it has a true negative rate near one

   ☐ a decision procedure with a small false positive rate must also have a small false discovery proportion

   Answer: False. True. False. [1 pt each]

Question 2 [10 points]. In the space below, explain how to apply a null hypothesis statistical test. Let x represent the observed data, and X hypothetical data. Use H_0 to denote the null hypothesis and H_1 the alternative. Let $s(x)$ denote the observed value of the test statistic and $S = s(X)$ denote hypothetical values. Let $p(s)$ the p-value given the observed value of the test statistic, s. Let α denote the level of the test.
Your explanation must:

- clearly define the p value as a probability
- clearly explain the conditions under which we reject the null, or fail to reject the null.
- explain which error rate is controlled by your procedure by completing the equation at the bottom of the page.

Procedure:

Answer: To run a NHST:

1. Observe x.
2. Compute $s(x)$.
3. Compute $p(s(x)) = \Pr(S \geq s(x))$ where $S = s(X)$. [3 pt]
4. Reject the null if $p(s(x)) \leq \alpha$, fail to reject otherwise. [2 pt]

Guarantee: This procedure controls the *false positive rate* [1.5 pt] in the sense:

$$\Pr(\text{reject}|H_0) \leq \alpha.$$

Give [1.5 pt] for the probability of rejection under the null ([0.75 pt] for rejection, [0.75 pt] for under the null), and [2 pt] for $\leq \alpha$.

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Question 3 [5 points]. Consider two simple hypotheses:

$$H_0 : X \sim f_0, \qquad H_1 : X \sim f_1$$

where f_0 and f_1 are different probability density functions.
In the space below, state the Neyman-Pearson lemma. Use the same notation from question 2 for the observed data, hypothetical data, test statistic, etc. Your statement must:

- Clearly define the test statistic suggested by the lemma in terms of f_0 and f_1.
- Clearly describe the rationale for using this statistic (e.g. what can we guarantee about the error rates?)

Answer: Any test that adopts the likelihood ratio:

$$\text{LR}(x) = f_1(x)/f_0(x)$$

or a monotonic injective transformation of the likelihood ratio, as its test statistic, is maximally sensitive among all tests with the same specificity (e.g. all tests with the same FPR control).

Give [1.5 pts] for selecting the likelihood ratio (or a transformation thereof) as the test statistic, [1.5 pts] for correctly defining the likelihood ratio, [1 pt] for maximizing sensitivity (equivalently power, TPR, or minimizing FNR), and [1 pt] for a controlled significance (equivalently, the same FPR or significance).

Do not require the statement about transformations of the statistic, or explicit definition of the threshold used for the test. Give full credit if the likelihood ratio is defined as null over alternative.

Question 4 [4 points]. Explain each error rate listed below in plain English, then suggest a multiple testing procedure that would control it. If you use a symbol in your definition, you must define it (e.g. $n_{0,0}$ is the number of true negative decisions).

- **False Discovery Rate:**

  Answer: The FDR is the expected proportion of discoveries (rejected nulls) that are false. [1 pt]

  Let $n_{r,d}$ denote the number of tests with decision d and reality r, so $n_{0,1}$ is the number of false discoveries and $n_{1,1}$ the number of true discoveries. Then:

  $$\text{FDR} = \mathbb{E}\left[\frac{n_{0,1}}{n_{0,1} + n_{1,1}}\right].$$

  To control the FDR, we should adopt the *Benjamini-Hochberg* [1 pt] procedure.

- **Family-Wise Error Rate:**

  Answer: The FWER is the probability that we make *at least one* false discovery across the whole family of tests [1 pt]. That is:

  $$\text{FWER} = \Pr(n_{1,0} \geq 1).$$

  To control the FWER, we should adopt the *Bonferroni* [1 pt] procedure.

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Question 5 [5 points]. Let Θ equal a randomly sampled unknown, X randomly sampled data, and $d(\cdot)$ a decision function that accepts data and returns a decision. Let $\ell(d(x), \theta)$ denote the loss associated with the decision $d(x)$ if $\Theta = \theta$. Then, define each notion of risk below:

- **Frequentist Risk:** $R_F(\theta) = \mathbb{E}_{X|\Theta=\theta}\left[\ell(d(X), \theta)\right]$ [1 pt]

- **Posterior Risk:** $R_p(x) = \mathbb{E}_{\Theta|X=x}\left[\ell(d(x), \Theta)\right]$ [1 pt]

- **Bayes Risk:** $R_B = \mathbb{E}_{\Theta,X}\left[\ell(d(X), \Theta)\right]$ [1 pt]

Then provide two equations, one relating Bayes risk to the Frequentist risk, and one relating the Bayes risk to the Posterior risk.

$$R_B = \mathbb{E}_{\Theta}\left[R_F(\Theta)\right] \text{ [1 pt] } = \mathbb{E}_X\left[R_p(X)\right] \text{ [1 pt].}$$

Rubric: For the first two definitions, give [0.25 pt] for the correct argument (θ or x), [0.5 pt] for averaging over the relevant variable, and [0.25 pt] for conditioning on the other.

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Part 2: *Complete 2 of the 4 problems. Indicate which problems you want graded below.*

Please Grade: ◯ Question 6, ◯ Question 7, ◯ Question 8, ◯ Question 9

If you fill in more than two bubbles, we will grade the first two problems you mark for grading.

Question 6 [15 points]. Suppose that $T \sim \text{Exponential}(\lambda)$ and wish to test

$$H_0 : \lambda = c \quad \text{vs} \quad H_1 : \lambda = b \times c$$

for some $b > 1$. Recall that, if $Y \sim \text{Exponential}(\mu)$ then $Y \geq 0$ with PDF and CDF:

$$f_Y(y) = \mu e^{-\mu y}, \quad F_Y(y) = 1 - e^{-\mu y}$$

(a) Compute the likelihood ratio, $\text{LR}(t)$, explicitly in terms of b and c.

Answer: Under the null $f_0(t) = ce^{-ct}$ and under the alternative $f_1(t) = bce^{-bct}$. Therefore, for $t \geq 0$:

$$\text{LR}(t) = \frac{f_1(t)}{f_0(t)} = \frac{bce^{-bct}}{ce^{-ct}} = be^{-(b-1)ct}. \text{ [3 pt]}$$

(b) Let $S = s(T) = \text{LR}(T)$. Let α be our false positive rate ($0 < \alpha < 1$). Compute the value of the threshold $s_*(\alpha)$ so that the FPR of the test is equal to α if we reject the null when $S \geq s_*(\alpha)$.

Answer: The FPR is $\Pr(S \geq s_*(\alpha)|H_0)$ [1.5 pt]. Since $b > 1$, $\text{LR}(t)$ is a decreasing function of t, so we can isolate T:

$$S \geq s_* \iff be^{-(b-1)cT} \geq s_* \iff T \leq t_* := -\frac{1}{(b-1)c} \log\left(\frac{s_*}{b}\right). \text{ [0.5 pt]}$$

Then, using the CDF under the null:

$$\alpha = \Pr(T \leq t_*|H_0) = 1 - e^{-ct_*} \implies t_* = -\frac{1}{c}\log(1-\alpha). \text{ [1.5 pt]}$$

Converting back to the likelihood ratio:

$$s_*(\alpha) = be^{-(b-1)ct_*} = be^{(b-1)\log(1-\alpha)} = b(1-\alpha)^{b-1}. \text{ [0.5 pt]}$$

Give full credit for other valid routes to the same solution, matching the assignment of credit used here as closely as possible ([1.5 pt] for controlling the correct probability, [1.5 pt] for working with the appropriate distribution function, [0.5 pt] for algebraic steps).

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(c) What is the true positive rate of this test? Your answer should be expressed in terms of b, c, and α.

Answer: The TPR is $\Pr(S \geq s_*(\alpha)|H_1) = \Pr(T \leq t_*|H_1)$ [1.5 pt]. Using the CDF under the alternative [1.5 pt]:

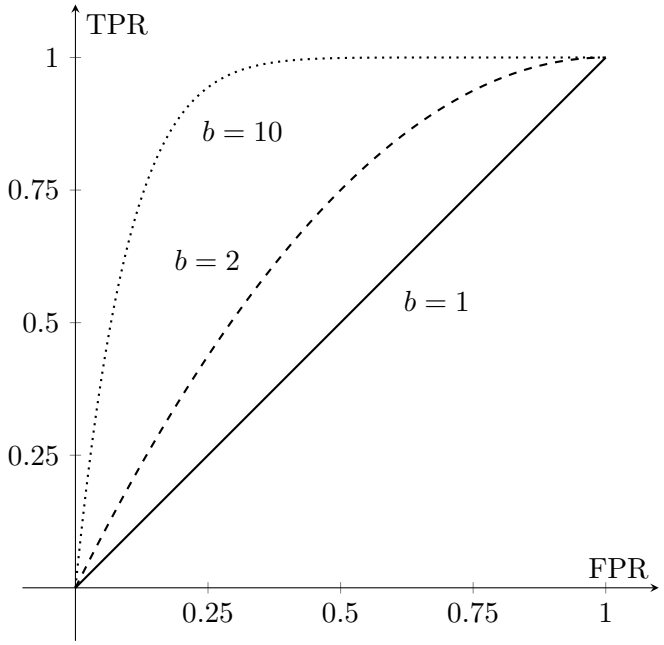
$$\text{TPR} = 1 - e^{-bct_*} = 1 - e^{b\log(1-\alpha)} = 1 - (1-\alpha)^b. \text{ [1 pt]}$$

(d) On the axes below, carefully draw a plot showing the TPR of the test as a function of desired FPR, α, for $b = 1$, $b = 2$, and $b = 10$. Clearly label each curve with the corresponding value of b. To receive full credit, it must be possible to correctly answer each question listed below using your plot (you do not need to answer the questions in writing, just provide sufficient illustration):

1. Is the TPR a monotonic function of the FPR? If so, is it increasing or decreasing in the FPR?
2. Is the TPR a concave or convex function of the FPR?
3. What is the TPR when $\alpha = 0$ and when $\alpha = 1$?
4. Holding α fixed, how does the TPR depend on b (increasing, decreasing, constant)?

Answer: Plot $\text{TPR}(\alpha) = 1 - (1-\alpha)^b$ for $b = 1$ (the diagonal, TPR $= \alpha$), $b = 2$, and $b = 10$. Give [1 pt] for each feature below that is correctly illustrated (deduct [0.5 pt] per unlabeled curve):

1. Increasing: raising α lowers the threshold, so the test makes more positive decisions.
2. Concave (for $b > 1$): the most diagnostic outcomes are admitted first, so each additional unit of α buys less TPR than the last.
3. TPR $= 0$ at $\alpha = 0$ (never reject) and TPR $= 1$ at $\alpha = 1$ (always reject).
4. Increasing in b: the larger b, the more distinct the two distributions, so the larger the TPR for any fixed FPR. At $b = 1$, $H_1 = H_0$, so the TPR equals α.



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Question 7 [15 points]. Suppose that we can observe $\{X_i\}_{i=1}^n$, drawn i.i.d. from either f_0 or f_1. To determine which distribution produced our data, we will run a hypothesis test. Treat the hypothesis, $X_i \sim f_0$ as the null.

(a) Show that the log-likelihood ratio is: $\log(\text{LR}(\vec{x})) = \sum_{i=1}^n \log(\text{LR}(x_i))$

Answer: Since the samples are independent, the joint density is a product of the marginals under either hypothesis [2 pts]:

$$\text{LR}(\vec{x}) = \frac{\prod_{i=1}^n f_1(x_i)}{\prod_{i=1}^n f_0(x_i)} = \prod_{i=1}^n \frac{f_1(x_i)}{f_0(x_i)} = \prod_{i=1}^n \text{LR}(x_i).$$

Then, since the logarithm of a product is the sum of the logs [1 pt]:

$$\log(\text{LR}(\vec{x})) = \sum_{i=1}^n \log(\text{LR}(x_i)).$$

(b) Consider the test statistic, $s(\vec{x}) = \frac{1}{n}\log(\text{LR}(\vec{x}))$. Is large s evidence in favor of, or against, the null? Justify your answer.

Answer: Against the null [1 pt].

The likelihood ratio is the likelihood of the alternative divided by the likelihood of the null, and s is a monotonically increasing function of the likelihood ratio. So, when s is large, the data is much more likely under the alternative than under the null [1 pt].

(c) Show that a test that decides for or against the null by comparing s to a threshold is maximally sensitive for any controlled significance.

Answer: The map $\text{LR} \mapsto \frac{1}{n}\log(\text{LR})$ is injective and monotonically increasing [1 pt], so:

$$s(\vec{x}) \geq s_* \iff \text{LR}(\vec{x}) \geq e^{ns_*}.$$

Therefore, the suggested test makes the same decisions as a likelihood ratio test, so has the same FPR and TPR. By the Neyman-Pearson lemma, the test that rejects when $\text{LR}(\vec{x})$ exceeds a threshold is the most powerful (maximally sensitive) test at any significance level α [1 pt]. Therefore, the test using s is also maximally sensitive at any controlled significance [1 pt].

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(d) Let $R_i = \log(\text{LR}(X_i))$, $\mathbb{E}[R_i|H_0] = \mu_0$, and $\text{SD}[R_i|H_0] = \sigma_0$. Assume that all of the expected values and standard deviations are finite. Then, show that, under the null, $S = s(\vec{X})$ converges in probability to μ_0, with deviations on the scale $\sigma_0/\sqrt{n}$, and is asymptotically Gaussian, in the limit as $n \to \infty$.

Answer: By part (a), $S = \frac{1}{n}\sum_{i=1}^{n} R_i$ is the sample average of the R_i [1 pt]. Since the X_i are i.i.d. under the null, the $R_i = \log(\text{LR}(X_i))$ are also i.i.d. under the null, with finite mean μ_0 and finite standard deviation σ_0. Therefore:

- By the weak law of large numbers, S converges in probability to $\mathbb{E}[R_i|H_0] = \mu_0$. [1 pt]
- By standard variance algebra, $\text{Var}[S|H_0] = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}[R_i|H_0] = \sigma_0^2/n$, so $\text{SD}[S|H_0] = \sigma_0/\sqrt{n}$. [1 pt]
- By the central limit theorem, $\sqrt{n}(S - \mu_0)/\sigma_0$ converges in distribution to a standard normal, so S is asymptotically Gaussian, $S \approx \mathcal{N}(\mu_0, \sigma_0^2/n)$. [1 pt]

(e) Let $s_*(\alpha)$ denote the decision threshold we compare S to when testing the null hypothesis f_0 at significance level α. Using part (d), identify the limiting value of $s_*(\alpha)$ and whether $s_*(\alpha)$ approaches its limit from above or below. Justify your answer.

Answer: Large S is evidence against the null, so we reject when $S \geq s_*(\alpha)$, where $s_*(\alpha)$ is chosen so that $\Pr(S \geq s_*(\alpha)|H_0) = \alpha$. By part (d), for large n, $S \approx \mathcal{N}(\mu_0, \sigma_0^2/n)$ under the null, so:

$$s_*(\alpha) \approx \mu_0 + \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}(1-\alpha)$$

where Φ is the standard normal CDF. [1.5 pt] Therefore, $s_*(\alpha)$ converges to μ_0 as $n \to \infty$ [1 pt].

For $\alpha < 1/2$ (any conventional significance level), $\Phi^{-1}(1-\alpha) > 0$, so $s_*(\alpha)$ approaches μ_0 *from above* [0.5 pt]. (For $\alpha > 1/2$ it would approach from below.)

Give up to [1.5 pt] for other valid justifications.

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Question 8 [15 points]. Suppose that you have census data from a random sample of the US adult population. Your data includes two numerical fields: `Age` and `Hours worked per week`. It also includes five categorical fields: Gender, Education (post high school, post college), Marriage status and whether the person's income is greater than $50,000. We will use this dataset to test the hypotheses of whether each of the categorical fields have any effect on the expectation of the numerical fields.

(a) Consider the following hypothesis:

H_0 : There is no difference in the average value of `numerical_col` between the two groups specified in `binary_col`.

H_1 : The average value of `numerical_col` is different for the two groups specified in `binary_col`.

Explain, in pseudocode, how you would estimate p-values under hypothesis H_0 for any pair of numerical and categorical fields.

Answer: Use a permutation test (A/B test). Under H_0, the group labels are exchangeable, so shuffling them produces draws from the null distribution of the test statistic.

1. Compute the observed statistic, the absolute difference in the mean of `numerical_col` between the two groups defined by `binary_col`:

$$t_{\text{obs}} = |\bar{x}_1 - \bar{x}_0|, \quad \bar{x}_j = \text{mean}(\texttt{numerical_col} \mid \texttt{binary_col} = j). \text{ [1 pt]}$$

2. For $k = 1, 2, \ldots, K$ (with K large, e.g. $K = 10{,}000$):
   (a) Randomly permute (shuffle without replacement) `binary_col`, holding `numerical_col` fixed. [1.5 pt]
   (b) Recompute the statistic on the shuffled data, t_k, in the same way as step 1. [1 pt]
3. Return the estimated p-value, the fraction of simulated statistics at least as extreme as the observed statistic: $p = \frac{1}{K} \sum_{k=1}^{K} \mathbf{1}(t_k \geq t_{\text{obs}})$. [1.5 pt]

Since H_1 is two-sided ("different"), the statistic should use the absolute difference in means. Deduct [0.5 pt] for a one-sided statistic without justification.

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(b) Suppose that, after 10,000 simulations, you receive a table of ten p-values, one for each possible combination of categorical and numerical columns:

		1	2
		Age	Hours per week
a	Post HS?	0.7	0.9
b	Post College?	0.4	0.15
c	Ever Married?	0	0.016
d	Gender	0.06	0.018
e	50K?	0.001	0.08

In your following answers, you may use H_{a1} to mean the "Post HS?" and "Age" hypothesis, H_{a2} to mean the "Post HS?" and "Hours per week" hypothesis, etc.

(i) Suppose we want to guarantee a Family-wise Error Rate (FWER) of 0.05. Given the p-values provided above, for which tests do we reject the null hypothesis?

Answer: Use the Bonferroni correction [1.5 pt]. With $m = 10$ tests, the threshold is $0.05/10 = 0.005$ [1 pt]. Only two p-values fall at or below 0.005, so we reject H_{c1} ($p = 0$) and H_{e1} ($p = 0.001$) [1 pt].

(ii) Suppose we want to guarantee a False Discovery Rate (FDR) of 0.05. Given the p-values provided above, for which tests do we reject the null hypothesis?

Answer: Use the Benjamini-Hochberg procedure [1.5 pt]. Sort the p-values and compare the k^{th} smallest to $k\alpha/m = 0.005k$ [1 pt]:

k	1	2	3	4	5	6	7	8	9	10
Hypothesis	$c1$	$e1$	$c2$	$d2$	$d1$	$e2$	$b2$	$b1$	$a1$	$a2$
$p_{(k)}$	0	0.001	0.016	0.018	0.06	0.08	0.15	0.4	0.7	0.9
$0.005k$	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040	0.045	0.050
$p_{(k)} \leq 0.005k$?	✓	✓	×	✓	×	×	×	×	×	×

The largest k with $p_{(k)} \leq k\alpha/m$ is $k = 4$ [1 pt], so we reject the four smallest: H_{c1}, H_{e1}, H_{c2}, and H_{d2} [1 pt].

Note: H_{c2} ($p = 0.016 > 0.015$) fails its own comparison but is still rejected, since BH rejects *every* hypothesis ranked at or below the largest passing k. Give [0.5 pt] of the final point to students who stop at the first failure and reject only H_{c1} and H_{e1}.

(c) How do the results from (b.i) and (b.ii) compare? Explain how and why these results are different.

Answer: Benjamini-Hochberg makes more discoveries (4 vs. 2), and its rejections include every Bonferroni rejection [1 pt]. Bonferroni controls the FWER, the probability of making *even one* false positive across all tests. Benjamini-Hochberg controls the FDR, the expected *proportion* of discoveries that are false. This tolerates some false positives, as long as they are accompanied by enough true positives, so its thresholds are larger and grow with the number of discoveries [1 pt].

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Question 9 [15 points]. Consider a medical diagnostic decision. For each person that undergoes testing, the null hypothesis is that they don't have the virus, and the alternative is that they do.

When making medical decisions, wrong diagnoses carry different costs. Deciding that a patient does not have the virus when in reality they do can have severe consequences: lack of treatment, risk of infecting others, and even premature death.

On the other hand, deciding that a patient has the virus when in reality they don't has consequences of that include distress, unnecessary treatment and costs of subsequent testing. These errors are typically considered less severe.

(a) Let's assume that we estimate the cost of making a false negative mistake to be k-times larger than the cost of a false positive. We can express that via the loss function shown below:

$$\begin{cases} \ell(D=0 \mid R=0) = \ell(D=1|R=1) = 0 \\ \ell(D=1 \mid R=0) = 1 \\ \ell(D=0 \mid R=1) = k \end{cases}$$

Fill in the missing value above.

Answer: k. [2 pt]

(b) Suppose that you run the test on 100 patients and observe the following outcomes:

	$D=0$	$D=1$
$R=0$	84	12
$R=1$	1	3

Compute the average loss incurred over these 100 patients.

Answer: There are 12 false positives (loss 1 each) and 1 false negative (loss k). Correct decisions incur no loss. So the average loss is:

$$\frac{12 \times 1 + 1 \times k}{100} = \frac{12+k}{100}. \text{ [2 pt]}$$

(c) Suppose that you can implement a test that, for a false positive rate α, achieves a true positive rate $\beta = 1 - (\alpha - 1)^2$. Suppose that the prevalence of the disease is 0.04. In the space below, express the Bayes risk as a function of $\alpha \in [0, 1]$ and k.

Answer: The Bayes risk is the expected loss, averaging over reality and the decision [1 pt]:

$$R_B = \ell(1|0)\Pr(R=0)\,\text{FPR} + \ell(0|1)\Pr(R=1)\,\text{FNR} = 0.96\,\alpha + 0.04\,k\,(1-\beta). \text{ [1 pt]}$$

Since $1 - \beta = (\alpha - 1)^2 = (1 - \alpha)^2$:

$$R_B(\alpha, k) = 0.96\,\alpha + 0.04\,k\,(1-\alpha)^2. \text{ [0.5 pt]}$$

(d) Let $\alpha_*(k)$ denote the choice of α that minimizes the Bayes risk as a function of k. (Note: you should be able to answer each question below without solving for α_* explicitly. If you want to solve for α_*, use the back of this page.)

(i) If k increases, then $\alpha_*(k)$ (◯ *increases*, ◯ *decreases*)

(ii) If the prevalence of the disease increases, then $\alpha_*(k)$ (◯ *increases*, ◯ *decreases*)

Answer: Increases. Increases. [1 pt each]

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(e) Given a prevalence of 0.04, write, find, as a function of α, the posterior probability that a patient has the disease if they test positive, $\Pr(R = 1|D = 1)$.

Answer: By Bayes' rule [1 pt]:

$$\begin{aligned}\Pr(R = 1|D = 1) &= \frac{\Pr(D = 1|R = 1)\Pr(R = 1)}{\Pr(D = 1|R = 1)\Pr(R = 1) + \Pr(D = 1|R = 0)\Pr(R = 0)} \\ &= \frac{0.04\,\beta}{0.04\,\beta + 0.96\,\alpha}. \text{ [1 pt]}\end{aligned}$$

Substituting $\beta = 1 - (1 - \alpha)^2 = \alpha(2 - \alpha)$ and simplifying (full credit without simplification):

$$\Pr(R = 1|D = 1) = \frac{0.04\,\alpha(2 - \alpha)}{0.04\,\alpha(2 - \alpha) + 0.96\,\alpha} = \frac{2 - \alpha}{26 - \alpha}. \text{ [0.5 pt]}$$

(f) Given that the test returns a positive result, a doctor can decide to start treatment ($T = 1$), or not ($T = 0$). Assume that $\ell(T|R) = \ell(D|R)$. Find the posterior risk of treating, or not treating, the patient, given that the test was positive:

- Posterior risk if untreated:

  Answer: Not treating incurs loss k if the patient is sick, and 0 otherwise:

  $$k\Pr(R = 1|D = 1) = k\,\frac{2 - \alpha}{26 - \alpha}. \text{ [1 pt]}$$

- Posterior risk if treated:

  Answer: Treating incurs loss 1 if the patient is healthy, and 0 otherwise:

  $$1 \cdot \Pr(R = 0|D = 1) = 1 - \frac{2 - \alpha}{26 - \alpha} = \frac{24}{26 - \alpha}. \text{ [1 pt]}$$

(g) Suppose that $\alpha = 0.5$. Given that a patient tests positive, should the doctor recommend treatment?

Answer: At $\alpha = 0.5$, $\Pr(R = 1|D = 1) = 1.5/25.5 = 1/17$. So the posterior risk is $k/17$ if untreated, and $16/17$ if treated [0.5 pt]. The doctor should recommend treatment if and only if it has the smaller posterior risk [1 pt]: treat if $k > 16$, do not treat if $k < 16$ (and either choice is fine if $k = 16$) [0.5 pt].