LECTURE EIGHTEEN

PROBABILITY INEQUALITIES

1. MARKOV
2. MOMENTS
3. CHERNOFF
4. HOEFFDING

MARKOV's Inequality

Qn: X : non-negative random variable.

$\mathbb{E}X = 1$

What can you say about $\mathbb{P}\{X \geq 2\}$.

e.g.: ① $X \equiv 1 \Rightarrow \mathbb{P}\{X \geq 2\} = 0$

② $X \sim \text{Poisson}(1)$

$$\mathbb{P}\{X \geq 2\}$$
$$= 1 - \mathbb{P}\{X = 0\} - \mathbb{P}\{X = 1\}$$
$$= 1 - \frac{1}{e} - \frac{1}{e} = 1 - \frac{2}{e} \approx 0.264$$

③ $X \sim \mathcal{E}xp(\text{mean} = 1)$

$$\mathbb{P}(X \geq 2) = \int_2^{\infty} e^{-x}\,dx = e^{-2} = 0.135$$

④ $X : \underset{0.5}{0} \;\&\; \underset{0.5}{2} \qquad \mathbb{E}X = 1$

$$\mathbb{P}(X \geq 2) = 0.5$$

__Markov's Inequality__:

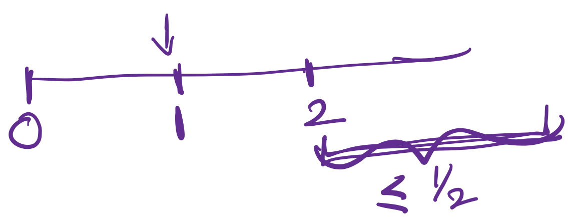
$X \geq 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}X}{t} \quad \text{for every } t > 0$$

__Special Case__: $\mathbb{E}X = 1$

$t = 2$

$$\mathbb{P}(X \geq 2) \leq \frac{1}{2} = 0.5$$



__Qn 2__: $X \geq 0$

$\mathbb{E}(X^2) = 1 \implies \mathbb{E}X \leq \underbrace{\sqrt{\mathbb{E}X^2}}_{1} = 1$

Then what can we say about

$$\mathbb{P}[X \geq 2] \leq \frac{\mathbb{E}X}{2} \leq \frac{\sqrt{\mathbb{E}X^2}}{2} = \frac{1}{2} = 0.5$$

$$\mathbb{P}[X \geq 2] = \mathbb{P}[X^2 \geq 4] \leq \frac{\mathbb{E}(X^2)}{4} = \frac{1}{4} = 0.25$$

Consequence of Markov's Inequality

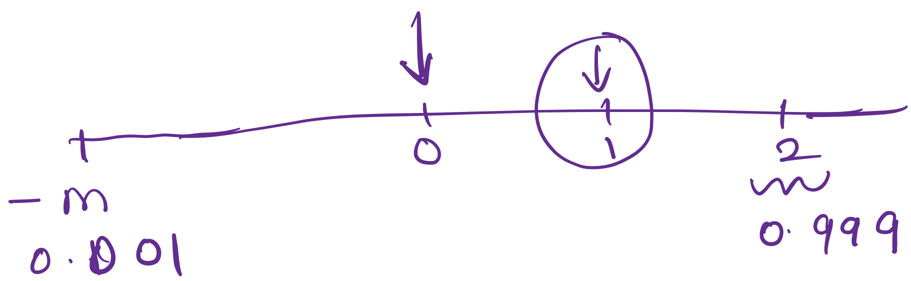
$\boxed{X \geq 0}$

$$\mathbb{P}[X \geq t] \leq \min\left[\frac{\mathbb{E}X}{t}, \frac{\mathbb{E}(X^2)}{t^2}\right]$$

Qn: X r.v that can take positive and negative values

$\mathbb{E}X = 1$

What can you say about $\mathbb{P}(X \geq 2)$?



$$(-m)\underbrace{(0.1)}_{(0.001)} + 2\underbrace{(0.9)}_{0.999} = 1$$

$X \geq 0$

$$\mathbb{P}[X \geq t] \leq \min\left[\frac{\mathbb{E}X}{t}, \frac{\mathbb{E}(X^2)}{t^2}\right]$$

$$\mathbb{P}[X \geq t] = \mathbb{P}[X^3 \geq t^3]$$
$$\leq \frac{\mathbb{E}(X^3)}{t^3}$$

<u>Moment-Bound:</u>

$$\left\{ \mathbb{P}[\underbrace{X}_{} \geq \underbrace{t}_{}] \leq \min_{r \geq 0} \left(\frac{\mathbb{E}(X^r)}{t^r} \right) \right\}$$

$$X \geq t \iff \underbrace{\frac{e^X}{1+e^X}}_{} \geq \frac{e^t}{1+e^t}$$

$$\boxed{\mathbb{P}(X \geq t)} = \mathbb{P}\left[\frac{e^X}{1+e^X} \geq \frac{e^t}{1+e^t} \right]$$
$$\left\{ \leq \boxed{\frac{\mathbb{E}\left(\frac{e^X}{1+e^X}\right)}{\left(\frac{e^t}{1+e^t}\right)}} \right\}$$

<u>CHERNOFF BOUND</u>

(Uses Exponentials with MARKOV)

X :

$$\mathbb{P}[X \geq t] \qquad \lambda \geq 0$$
$$= \mathbb{P}[e^{\lambda X} \geq e^{\lambda t}]$$

MARKOV

$$\leq \frac{\mathbb{E}(e^{\lambda X})}{e^{\lambda t}}$$

$$\mathbb{P}[X \geq t] \leq \min_{\lambda \geq 0} \left(\frac{\mathbb{E}(e^{\lambda X})}{e^{\lambda t}} \right)$$

$\longrightarrow$ CHERNOFF BOUND.

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① MOMENT BOUND:

$$\mathbb{P}(X \geq t) \leq \min_{r \geq 0} \left[\frac{\mathbb{E} X^r}{t^r} \right]$$

② CHERNOFF BOUND:

$$\mathbb{P}(X \geq t) \leq \underbrace{\min_{\lambda \geq 0} \left[\frac{\mathbb{E} e^{\lambda X}}{e^{\lambda t}} \right]}$$

MGF

② $\leq$ ①

FACT: $X \geq 0$ MOMENT BOUND IS ALWAYS BETTER THAN CHERNOFF.

$$\frac{\mathbb{E}\, e^{\lambda X}}{e^{\lambda t}} = \frac{\mathbb{E}\left[\sum_{r=0}^{\infty} \frac{(\lambda X)^r}{r!} \right]}{\sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!}}$$

$$= \frac{\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} (\mathbb{E} X^r)}{\ldots \lambda^r t^r}$$

$$\sum_{r=0}^{\infty} \frac{\lambda^r t^r}{r!}$$

$$= \frac{\displaystyle\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} t^r \left(\frac{\mathbb{E} X^r}{t^r}\right)}{\displaystyle\sum_{r=0}^{\infty} \frac{\lambda^r t^r}{r!}}$$

$$\geq \frac{\displaystyle\min_{r \geq 0} \frac{\mathbb{E}(X^r)}{t^r} \cancel{\sum_{r=0}^{\infty} \frac{\lambda^r t^r}{r!}}}{\displaystyle\cancel{\sum_{r=0}^{\infty} \frac{\lambda^r t^r}{r!}}}$$

<u>NOTE</u>: CHERNOFF BOUND:

$$\mathbb{P}(X \geq t) \leq \min_{\lambda \geq 0} \frac{\mathbb{E} e^{\lambda X}}{e^{\lambda t}}$$

IS VALID FOR ALL X (NOT NEC. NONNEGATIVE)

<u>EXAMPLE ONE</u>:

$X \sim \text{Bin}(n, p)$

$n = 3000, \; p = 0.5$

$$\mathbb{P}(X \geq 2000)$$

$$= \sum_{k=2000}^{3000} \mathbb{P}(X = k)$$

$$= \sum_{k=2000}^{3000} \binom{3000}{k} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{3000-k}$$

$$= \sum_{k=2000}^{3000} \underbrace{\binom{3000}{k} 2^{-3000}}\Big] \Big\}$$

$$\approx \underbrace{5.045 \times 10^{-76}}$$

$$\mathbb{P}[X \geq t] \leq \min_{\lambda \geq 0} \frac{\mathbb{E}\, e^{\lambda X}}{e^{\lambda t}}$$

$$\mathbb{E}\, e^{\lambda X} = \mathbb{E}\, e^{\lambda(X_1 + \cdots + X_n)}$$

$X = X_1 + \cdots + X_n$

$X_i \overset{iid}{\sim} \mathrm{Ber}(p)$

$$= \mathbb{E}\left[e^{\lambda X_1} e^{\lambda X_2} \cdots e^{\lambda X_n}\right]$$

$$= \left(\mathbb{E}\, e^{\lambda X_1}\right)\left(\mathbb{E}\, e^{\lambda X_2}\right) \cdots \left[\mathbb{E}\left(e^{\lambda X_n}\right)\right]$$

$$= \left(\mathbb{E}\, e^{\lambda X_1}\right)^{n}$$

$$= \left(1 - p + p e^{\lambda}\right)^{n}$$

$$\mathbb{E}\, e^{\lambda X_1} = \underbrace{e^{\lambda(0)}}_{1} \underbrace{\mathbb{P}(X_1 = 0)}_{1-p} + \underbrace{e^{\lambda(1)}}_{e^{\lambda}} \underbrace{\mathbb{P}(X_1 = 1)}_{p}$$

Chernoff Bound:

$$\mathbb{P}(X \geq t) \leq \min_{\lambda \geq 0} \frac{\mathbb{E}\, e^{\lambda X}}{e^{\lambda t}}$$

$$= \min_{\lambda \geq 0} \frac{(1-p+pe^{\lambda})^n}{e^{\lambda t}}$$

$$= \min_{\lambda \geq 0} \exp\left[\underbrace{-\lambda t + n \log(1-p+pe^{\lambda})}\right]$$

$$-t + \frac{n\,(pe^{\lambda})}{1-p+pe^{\lambda}} = 0$$

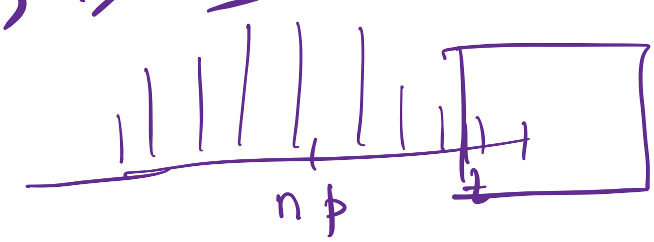
$$\Rightarrow e^{\lambda} = \frac{t(1-p)}{(n-t)\,p}$$

$$\lambda = \log \frac{t(1-p)}{(n-t)\,p}$$

Check: $\lambda \geq 0$ if $\frac{t}{n} \geq p$

Bound Chernoff: $t \geq np$

$$\mathbb{P}[\mathrm{Bin}(n,p) \geq t]$$



$$\downarrow$$

$$\leq \exp\left[-n\, \underbrace{D\left(\tfrac{t}{n}, p\right)}_{D\left(\frac{2}{3}, \frac{1}{2}\right)}\right]$$

$$D(\rho, p) = \rho \log \frac{\rho}{p} + (1-\rho) \log \frac{1-\rho}{1-p}$$

Eg: $n = 3000$, $p = 0.5$, $t = 2000$

$$\exp\left[-n\left(\left(\frac{2}{3}\right) \log \frac{(2/3)}{0.5} + \left(1-\frac{2}{3}\right) \log \frac{1-2/3}{0.5}\right)\right]$$

$\simeq 1.636 \times 10^{-74} \rightarrow$ CHERNOFF

$5.045 \times 10^{-76} \rightarrow$ ACTUAL

$$\mathbb{P}[X \geq t]$$
$$= \mathbb{P}\left[\frac{X}{n} \geq \frac{t}{n}\right]$$
$$\leq \boxed{\exp\left[-n \times D\left(\frac{t}{n}, p\right)\right]}$$

EXAMPLE TWO: GAUSSIAN

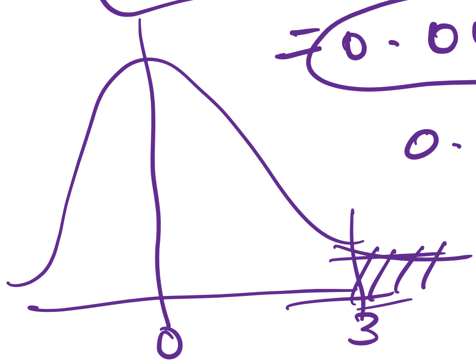
$X \sim N(0,1)$

What is $\boxed{\mathbb{P}(X \geq 3)}$

0.0015

$= 0.00135$

0.011



CHERNOFF:

$t = 3$

$$\min_{\lambda \geq 0} \frac{\mathbb{E}\, e^{\lambda X}}{e^{\lambda t}}$$

$$\mathbb{E}\, e^{\lambda X} = \int_{-\infty}^{\infty} e^{\lambda x} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx$$

$X \sim N(0,1)$

$$\overset{?}{=} e^{\frac{1}{2}\lambda^2}$$

CHECK

Chernoff: $$\min_{\lambda \geq 0} \frac{e^{\frac{1}{2}\lambda^2}}{e^{\lambda t}}$$

$$= \min_{\lambda \geq 0} \exp\left[-\lambda t + \frac{1}{2}\lambda^2\right]$$

$$\frac{d}{d\lambda}\left(-\lambda t + \frac{1}{2}\lambda^2\right) = -t + \lambda = 0$$

$$\Rightarrow \lambda = t$$

Conclude:

$$\mathbb{P}[X \geq t] \leq e^{-\frac{1}{2}t^2}$$

$$\parallel$$

$$\int_t^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$t = 3$

$\exp(-4.5)$

$= 0.011$

SUMMARY

① MARKOV $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$

$X \geq 0$
$t > 0$

② $\mathbb{P}(\underbrace{X} \geq t)$ $e^{\lambda X}$

CHERNOFF:

$$\mathbb{P}(X \geq t) \leq \min_{\lambda \geq 0} \frac{\mathbb{E} e^{\lambda X}}{e^{\lambda t}}$$

HOEFFDING INEQUALITY

$$\mathbb{P}[\text{Bin}(n, p) \geq t] \quad t \geq np$$

CHERNOFF

$$\leq \exp\left[-n \; D\left(\frac{t}{n}, p\right)\right]$$

HOEFFDING:

$$\mathbb{P}[\text{Bin}(n,p) \geq t]$$
$$\leq \exp\left[-n \, 2\left(\frac{t}{n} - p\right)^2\right]$$

SIMPLER EXPRESSION

Eg: $n = 3000$, $p = 0.5$, $t = 2000$

Actual : 5.045×10^{-76}

CHERNOFF: 1.636×10^{-74}

HOEFFDING: 4.145×10^{-73}

GENERAL HOEFFDING

X_i iid $\quad X_i, \; i = 1, \dots, n$ independent

$EX_i = \mu \qquad a \le X_i \le b \qquad$ X_i's are BOUNDED RVs

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i \ge t\right] \le \underbrace{\exp\left[-2n\left(\frac{t}{n} - p\right)^2\right]}_{\text{Binomial}}$$

$$\boxed{\exp\left[\frac{-2n\left(\frac{t}{n} - \mu\right)^2}{(b-a)^2}\right]}$$