

LECTURE SEVEN

PLAN FOR TODAY

- ① Basic facts about Beta Densities
- ② Previous Lecture Recap
- ③ Beta priors & posteriors (Bernoulli or Binomial likelihoods)
- ④ Real Data Application

Beta Density

$$f(u) = \frac{\mathbb{I}\{0 \leq u \leq 1\} u^{a-1} (1-u)^{b-1}}{\int_0^1 v^{a-1} (1-v)^{b-1} dv}$$

$$\text{Beta}(3, 5) \quad u^2 (1-u)^4$$

- ① $a=1, b=1$
 $\text{Beta}(1, 1) = \text{Uniform } \mathbb{I}\{0 \leq u \leq 1\}$
- ② Mean corresponding to $\text{Beta}(a, b)$
 $= \frac{a}{a+b}$

$$\text{Beta}(1, 4) \left(\text{mean} = \frac{1}{1+4} = 0.2 \right)$$

$$\text{Beta}(4, 1) \left(\text{mean} = \frac{4}{4+1} = 0.8 \right)$$

③ Variance corresponding to $\text{Beta}(a, b)$

$$= \frac{a}{a+b} \times \frac{b}{a+b} \times \frac{1}{a+b+1}$$

Corollary: $a+b$ large \Rightarrow variance small
 \Downarrow
skinny Beta

LAST LECTURE

Quality of a product
based on reviews: Pos, Neg
 θ $\text{Pos} + \text{Neg} = \text{Tot}$

$$\text{Proportion of positive reviews} = \frac{\text{Pos}}{\text{Pos} + \text{Neg}}$$

\rightarrow ① Naive Estimate of θ

② Proportion

③ Naive Proportion

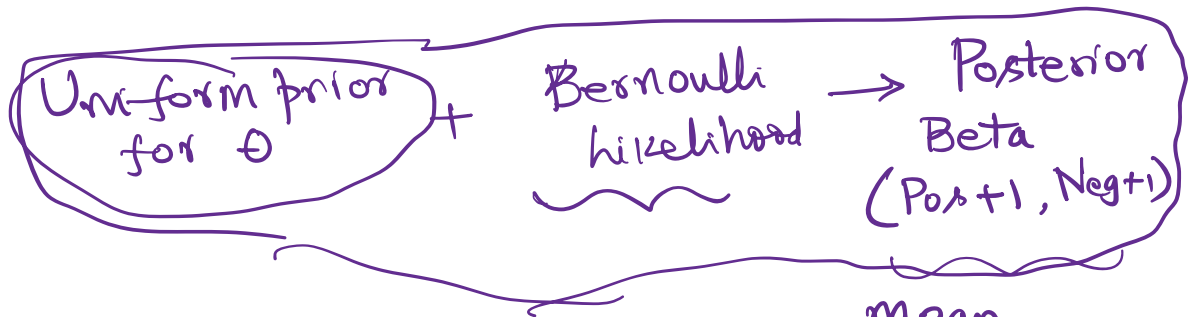
④ MLE

⑤ Frequentist Estimate

likelihood: $TP(\text{observing the data} | \theta = u)$

$$= \theta^{\text{Pos}} (1 - \theta)^{\text{Neg}}$$

Prior: Uniform prior.



$$\text{mean} = \frac{\text{Pos} + 1}{\text{Tot} + 2}$$

$$\text{Uniform} = \text{Beta}(1, 1)$$

$$\text{Beta}(1, 1) \text{ prior} + \text{Bernoulli likelihood} \Rightarrow \text{Beta}(1 + \text{Pos}, 1 + \text{Neg}) \text{ Posterior}$$

$$\text{Beta}(a, b) \text{ prior} + \text{Bernoulli likelihood} \Rightarrow \text{Beta}(a + \text{Pos}, b + \text{Neg})$$

Eg: Very rare for the quality to be below 0.8. Typically quality is 0.9.

$$\frac{a}{a+b} = 0.9 \Rightarrow a = 9b$$

$$\text{standard deviation} = 0.05$$

$$\sqrt{\frac{a}{a+b} \frac{b}{a+b} \frac{1}{a+b+1}} = 0.05$$

$$\sqrt{\frac{0.9 \times 0.1}{a+b+1}} = 0.05$$

$$\Rightarrow a+b+1 = \frac{0.09}{0.0025} = \frac{900}{25} = 36$$

$$\left. \begin{array}{l} a+b = 35 \\ a = 9b \end{array} \right\} \text{Solve for } a \text{ \& } b.$$

$$\begin{array}{l} b = 3.5 \\ a = 31.5 \end{array}$$

Beta (31.5, 3.5)

↓

Beta (31.5 + Pos, 3.5 + Neg)

$$u^{a-1} (1-u)^{b-1} \times u^{\text{Pos}} (1-u)^{\text{Neg}}$$

Comparison Between Frequentist & Bayes Estimate

$$\theta \rightarrow \frac{\text{Pos}}{\text{Pos} + \text{Neg}} \quad (\text{Frequentist})$$

↓
Posterior

$$\text{Beta}(a + \text{Pos}, b + \text{Neg})$$

$$\begin{aligned} \text{Bayes Estimate} &= \text{Posterior Mean} \\ &= \frac{a + \text{Pos}}{a + b + \text{Pos} + \text{Neg}} \end{aligned}$$

Qn: When are $\frac{Pos}{Tot}$ & $\frac{a+Pos}{a+b+Tot}$ same or different?

$$\frac{a+Pos}{a+b+Tot} = \left(\frac{a+b}{a+b+Tot}\right) \frac{a}{a+b} + \left(\frac{Tot}{a+b+Tot}\right) \frac{Pos}{Tot}$$