$\boxed{\text{Lecture Six}}$ DATA 102

Overview Of Bayesian Statistics

Question: What is Bayesian Statistics?

Bayesian Statistics = PROBABILITY THEORY

Example

I want to buy a MICROWAVE.

There are two options: Microwave A
Microwave B

A : 3 positive reviews, 0 negative reviews

B : 19 positive reviews, 1 negative review

Qn: Which should I buy?

$\mathbb{P}$[B is better than A | observed data on reviews]

θ_A = quality of Microwave A

θ_B = quality of Microwave B

Posterior Probability

$$\mathbb{P}\left[\theta_B > \theta_A \,\middle|\, +_A = 3, -_A = 0, +_B = 19, -_B = 1\right]$$

θ_A, θ_B : parameters.

Simpler Question: (a) Infer θ_A from reviews on A

(b) Infer θ_B from reviews on B.

Posterior of θ_A given $+_A = 3$, $-_A = 0$

For this calculation, we need

(1) Prior : Distribution of θ_A before observing any review data.

$$\theta_A \in [0, 1]$$

0 : lowest possible quality

1 : highest possible quality

$$\theta_A \in \{0, 0.01, 0.02, 0.03, \dots, 0.98, 0.99, 1\}$$

$$\theta_A \sim \text{uniform}\{0, 0.01, \dots, 0.99, 1\}$$

$$\boxed{\mathbb{P}\{\theta_A = u\} = \frac{1}{101}}$$

(2) Likelihood :

$$\mathbb{P}\{\text{observed data } +_A = 3, -_A = 0 \mid \theta_A = u\}$$

$$= u^3$$

Interpretations for θ_A

① θ_A : probability of a positive review.

② θ_A : Population proportion of positive reviews

③ θ_A : Proportion of usage instances where the user is satisfied

Posterior for θ_A

$$\mathbb{P}\left\{ \theta_A = u \,\middle|\, \begin{matrix}\text{observed}\\ \text{review data}\end{matrix} \right]$$

$$= \frac{\mathbb{P}\left[\theta_A = u\right] \times \mathbb{P}\left[\begin{matrix}\text{observed}\\ \text{review data}\\ \text{for } A\end{matrix} \,\middle|\, \theta_A = u\right]}{\mathbb{P}\left(\begin{matrix}\text{observed review}\\ \text{data for } A\end{matrix}\right)}$$

Denominator: marginal probability of the observed data

$$\mathbb{P}\left(\begin{matrix}\text{observed}\\ \text{review data}\\ \text{for } A\end{matrix}\right)$$

$$= \sum_{v} \mathbb{P}\left(\begin{matrix}\text{observed}\\ \text{review data}\\ \text{for } A\end{matrix} \,\middle|\, \theta_A = v\right) \mathbb{P}(\theta_A = v)$$

$$\mathbb{P}\left(\theta_A = u \,\middle|\, \begin{matrix}\text{observed}\\ \text{review}\\ \text{data}\end{matrix}\right)$$

$$= \frac{\mathbb{P}(\theta_A = u) \times \mathbb{P}\left(\substack{\text{observed}\\ \text{data}\\ \text{for } A} \,\middle|\, \theta_A = u\right)}{\sum_v \mathbb{P}(\theta_A = v) \times \mathbb{P}\left(\substack{\text{observed}\\ \text{data}\\ \text{for } A} \,\middle|\, \theta_A = v\right)}$$

$$= \frac{\frac{1}{101} \times u^3}{\sum_v \frac{1}{101} \times v^3}$$

$$\mathbb{P}\left(\theta_B = u \,\middle|\, \substack{\text{review}\\ \text{data}\\ \text{for } B}\right)$$

$$= \frac{\mathbb{P}(\theta_B = u) \times \mathbb{P}\left(\substack{\text{observed}\\ \text{reviews}\\ \text{for } B} \,\middle|\, \theta_B = u\right)}{\sum_v \mathbb{P}(\theta_B = v) \times \mathbb{P}\left(\substack{\text{observed}\\ \text{reviews}\\ \text{for } B} \,\middle|\, \theta_B = v\right)}$$

$$= \frac{\frac{1}{101} \times u^{19}(1-u)}{\sum_v \frac{1}{101} \times v^{19}(1-v)}$$

$$\mathbb{P}\left(\theta_A \leq \theta_B \,\middle|\, \substack{\text{observed}\\ \text{data}\\ \text{for both}}\right)$$

$$= \sum_{\substack{u_A, u_B \,:\, u_A \leq u_B\\ \in \{0, 0.01, 0.02, \ldots, 1\}}} \mathbb{P}\left(\theta_A = u_A, \theta_B = u_B \,\middle|\, \substack{\text{observed}\\ \text{data}\\ \text{for both}}\right)$$

$$= \sum_{u_A, u_B : u_A \le u_B} \underbrace{\mathbb{P}(\Theta_A = u_A \mid \text{reviews for } A)\,\mathbb{P}(\Theta_B = u_B \mid \text{reviews for } B)}$$

$$= \sum_{u_A, u_B : u_A \le u_B} \left(\frac{\frac{1}{101} \times u_A^3}{\sum_{v_A} \frac{1}{101} \times v_A^3} \right) \times \left(\frac{\frac{1}{101} \times u_B^{19} \times (1-u_B)}{\sum_{v_B} \frac{1}{101} v_B^{19} (1-v_B)} \right)$$

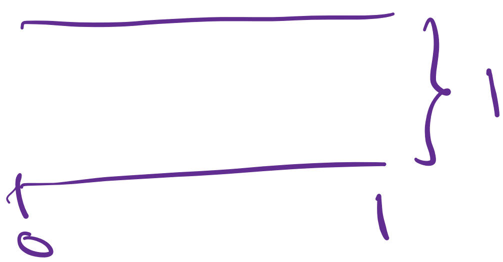
$$= 0.69$$

Continuous Priors

Prior:

$$\Theta_A \sim \text{Unif}\{0, 0.01, 0.02, \ldots, 0.98, 0.99, 1\}$$

$$\sim \text{Unif}[0, 1]$$



$$\mathbb{P}\{\Theta_A = u\} = \frac{1}{101}$$

$$\boxed{f_{\Theta_A}(u) = \mathbb{I}\{0 \le u \le 1\}}$$

Likelihood:

$$\mathbb{P}(\text{observed reviews for } A \mid \Theta_A = u) = u^3$$

Posterior

$$\mathbb{P}(\Theta_A = u \mid \text{observed data})$$

$$= \frac{\mathbb{P}(\Theta_A = u) \times \mathbb{P}(\text{observed} \mid \Theta_A = u)}{\sum_v \mathbb{P}(\Theta_A = v) \times \mathbb{P}(\text{observed} \mid \Theta_A = v)}$$

$$\boxed{f_{\Theta_A \mid \text{observed data}}(u) = \frac{f_{\Theta_A}(u) \times \mathbb{P}(\text{observed data} \mid \Theta_A = u)}{\int f_{\Theta_A}(v) \times \mathbb{P}(\text{observed data} \mid \Theta_A = v)\, dv}}$$

$$= \frac{I\{0 \le u \le 1\} \times u^3}{\int I\{0 \le v \le 1\} \times v^3\, dv}$$

$$= \boxed{4u^3\, I\{0 \le u \le 1\}}$$ Beta(4,1) density

Recall: Beta densities

$$\frac{I\{0 \le u \le 1\}\, u^{a-1}(1-u)^{b-1}}{\int_0^1 v^{a-1}(1-v)^{b-1}\, dv}$$

Θ_B $$f_{\Theta_B \mid \text{data}}(u) = \frac{f_{\Theta_B}(u)\, \mathbb{P}(\text{data} \mid \Theta_B = u)}{\int f_{\Theta_B}(v)\, \mathbb{P}(\text{data} \mid \Theta_B = v)\, dv}$$

$$= \frac{1 \times u^{19}(1-u) I\{0 \le u \le 1\}}{\int_0^1 v^{19}(1-v)\,dv}$$

$$= \frac{u^{19}(1-u) I\{0 \le u \le 1\}}{\frac{1}{20} - \frac{1}{21}}$$

$$= 420\, u^{19}(1-u) I\{0 \le u \le 1\}$$

$$\boxed{\text{Beta}(20, 2)}$$

Home: $\mathbb{P}(\theta_A < \theta_B \mid \text{data}) = 0.7$