Lecture 22: Markov Decision Processes

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Complex Decision-Making

Previous lectures have explored several themes:

- Decision-making (e.g. FDR, bandits)
- Time dynamics and statefulness (e.g. Markov models)
- Value of information (e.g. multi-armed bandits)

We will combine all of these with

1. *Markov decision processes* (stateful decision-making), and
2. *reinforcement learning* (stateful decision-making with uncertainty).

Roadmap

- Review: dynamic programming
- Markov decision processes
    - Bellman equations
    - Solution via dynamic programming
- Reinforcement learning (next lecture)

Dynamic programming warm-up: Fibonacci

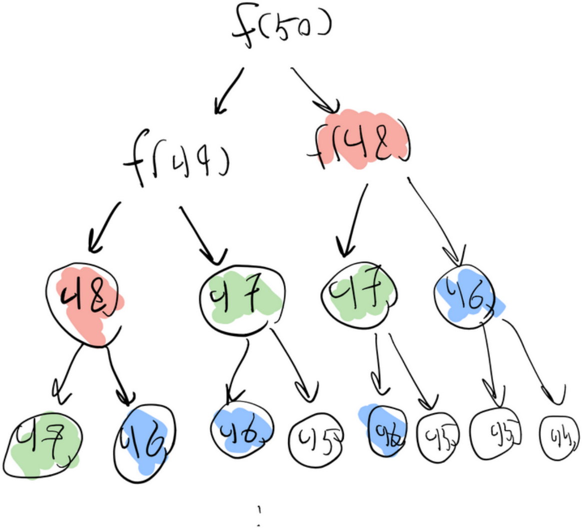
Fibonacci sequence: $F_n = F_{n-1} + F_{n-2}$ ($F_0 = 0$, $F_1 = 1$)

Recursive function:

```
def fib(n):
  if n <= 1:
    return n
  else:
    return fib(n-1) + fib(n-2)
```

What happens if we call `fib(50)`?

Exponential blow-up



Solution 1: Memoization

Remember answers in a `dict`:

```
memo_dict = dict()
def fib(n):
    if n in memo_dict.keys():
        return memo_dict[n]
    elif n <= 1:
        ans = n
    else:
        ans = fib(n-1) + fib(n-2)
    memo_dict[n] = ans
    return ans
```

Solution 1: Memoization

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  memo_dict[n] = ans
  return ans
```

- Can use decorators for slick code
- Slow (dict lookup each time)

Solution 2: Dynamic Programming

Can replace with for loop if we do things in right oder:

```
import numpy as np
n_max = 50
fibs = np.array(n_max)
fibs[0], fibs[1] = 0, 1
for n in range(2, n_max):
  fibs[n] = fibs[n-1] + fibs[n-2]
```

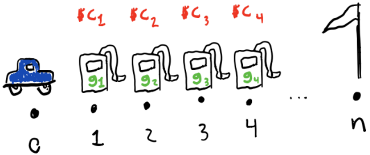
Solution 2: Dynamic Programming

Can replace with for loop if we do things in right oder:

```
import numpy as np
n_max = 50
fibs = np.array(n_max)
fibs[0], fibs[1] = 0, 1
for n in range(2, n_max):
  fibs[n] = fibs[n-1] + fibs[n-2]
```

- Pro: fast, low-memory
- Con: more thinking; need to find linear structure

Harder example: car and gas stations

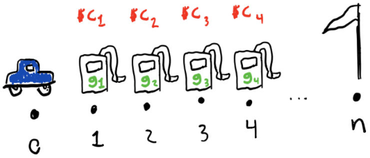


- Locations $0, \ldots, n$
- Car starts at location 0, wants to get to location n
- Each location i: gas station selling g_i units of gas at c_i dollars per unit
- 1 unit of gas to move 1 unit right

Challenge:
How much gas should we buy at each location to minimize total cost?

Solution via recursion

- State: (location, gas left in tank)
- Define $f(\text{loc}, \text{gas}) =$ minimum cost to get to end given current state ("cost-to-go")
- Two options: buy 1 unit of gas (stay where we are), or go forward



Solution via recursion

- State: (location, gas left in tank)
- Define $f(\mathrm{loc}, \mathrm{gas}) =$ minimum cost to get to end given current state ("cost-to-go")
- Two options: buy 1 unit of gas (stay where we are), or go forward

```
def f(loc, gas):
  if loc == n:
    return 0
  if gas < 0:
    return -np.inf
  cost1 = f(loc, gas+1) + price[loc]
  cost2 = f(loc+1, gas - 1)
  return min(cost1, cost2)
```

Solution via dynamic programming

```
f = np.zeros(shape=(n+1,n+1))
for loc in range(n-1, -1, -1):
  for gas in range(n, -1, -1):
    cost1 = f[loc, gas+1] + price[loc]
    cost2 = f[loc+1, gas-1]
    f[loc, gas] = min(cost1, cost2)
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Solution via dynamic programming

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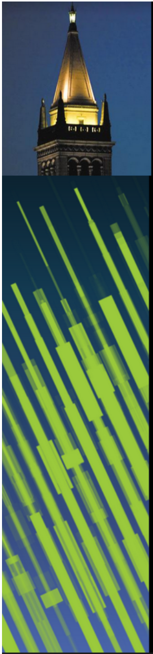
```

- Gas station problem is special case of *Markov decision process*
- Will define these next and see how to formulate a general dynamic programming solution



DS 102: Data, Inference, and Decisions

Lecture 23: Markov Decision Processes

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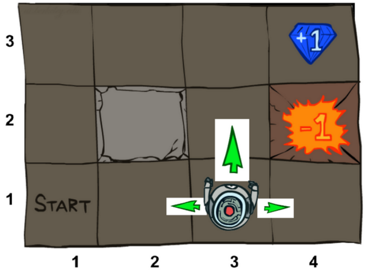
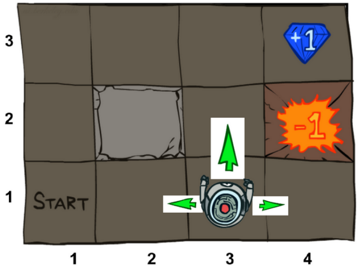
Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function $T(s, a, s')$
    - Probability that a from s leads to s', i.e., $P(s' | s, a)$
    - Also called the model or the dynamics
  - A reward function $R(s, a, s')$
    - Sometimes just $R(s)$ or $R(s')$
  - A start state
  - Maybe a terminal state



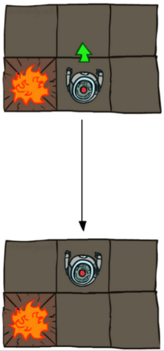
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

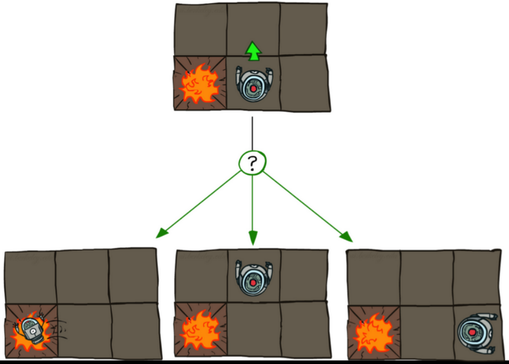


Grid World Actions

Deterministic Grid World



Stochastic Grid World



What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent

- For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

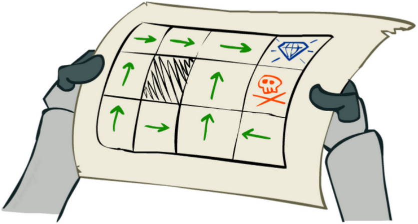
- This is just like search, where the successor function could only depend on the current state (not the history)

Andrey Markov
(1856-1922)

Policies

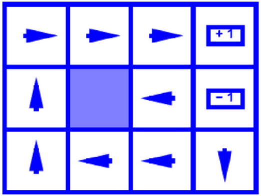
- In deterministic single-agent search problems, we want an optimal plan, or sequence of actions, from start to a goal

- For MDPs, we want an optimal
  policy $\pi^*: S \rightarrow A$
  - A policy π gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent

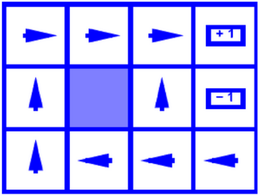


Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

Optimal Policies



R(s) = -0.01



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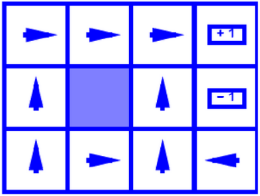
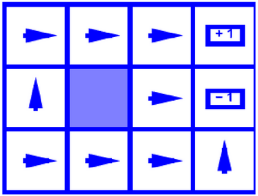


R(s) = -0.4



R(s) = -2.0

Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially

1

Worth Now

γ

Worth Next Step

γ^2

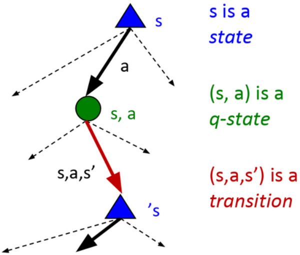
Worth In Two Steps

Optimal Quantities

- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally

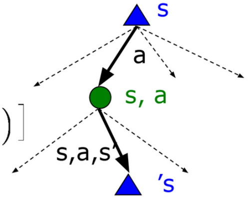
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



Values of States

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$



$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s') \right]$$

Solving the recursion

Recursion for V^* is circular:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

Solving the recursion

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$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- Not a problem for gas stations because states were totally ordered
- Can't assume this in general
- Solution: add a time component

$$V^*(s,t) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s',t-1)],$$
$$V^*(s,0) = 0$$

Solving the recursion

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$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

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$$V^*(s,t) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s',t-1)],$$

$$V^*(s,0) = 0$$

- Time t creates total ordering!
- Can recover $V^*(s)$ by taking $t \to \infty$

Value learning via dynamic programming

```
V = np.zeros(shape=(num_states, t_max))
for t in range(1, t_max):

  for s in range(num_states):
    V[s, t] = max([sum([T(s, a, s2) * (R(s, a, s2)
                          + gamma * V[s2, t-1])
                        for s2 in num_states])
                   for a in num_actions])
```

Value learning via dynamic programming

Can save memory with "sliding window" trick:

```
V = np.zeros(shape=(num_states, t_max))
for t in range(1, t_max):

  for s in range(num_states):
    V[s, t] = max([sum([T(s, a, s2) * (R(s, a, s2)
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                        for s2 in num_states])
                   for a in num_actions])
```

Value learning via dynamic programming

Can save memory with “sliding window” trick:

```
V = np.zeros(num_states)
for t in range(1, t_max):
  V_old = np.copy(V)
  for s in range(num_states):
    V[s]    = max([sum([T(s, a, s2) * (R(s, a, s2)
                         + gamma * V_old[s2])
                         for s2 in num_states])
                    for a in num_actions])
```

Exploiting monotonicity

Since updates monotonically approach V^*, can update in place:

```
V = np.zeros(num_states)
for t in range(1, t_max):

  for s in range(num_states):
    V[s]    = max([sum([T(s, a, s2) * (R(s, a, s2)
                        + gamma * V[s2])
                      for s2 in num_states])
                    for a in num_actions])
```

Recap

- Defined Markov decision process:
  - states, actions, (stochastic) transitions, rewards
- Recursion (Bellman equations)
- Efficient solution via dynamic programming
- Even more efficient solution exploiting monotonicity (in-place updates)
- Next lecture: what if transitions need to be learned? (RL)