Lecture 11: Frequentist Regression and Bootstrap

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Announcements

- Midterm next Tuesday!
- Review session this weekend (time TBD)
- Probably remote, with recording uploaded

Pitfalls of Bayes

Peril of Bayesian thinking: at the mercy of your model

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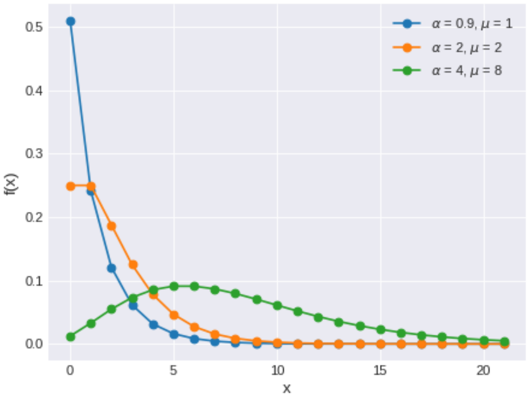
Typical fix: negative binomial distribution

$$p_{\mu,\alpha}(k) \propto \binom{k+\alpha-1}{k} \left(\frac{\mu}{\mu+\alpha}\right)^k$$

Mean μ, overdispersion α (variance $\mu \cdot (1 + \mu/\alpha)$)

Negative binomial plots



[Credit: PyMC3 docs]

Negative binomial regression on turbine data

[Jupyter demo]

Logistic regression revisited

Recall loss function for logistic regression: $L(\theta) = \frac{1}{n}\sum_{i=1}^{n} \ell(x_i, y_i; \beta)$, where

$$\ell(x, y; \beta) = -y \log \sigma(\beta^\top x) - (1-y) \log(1 - \sigma(\beta^\top x))$$

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- Exponentiate to make positive, normalize to add up to 1
- Generalization: softmax $\exp(z_j)/\sum_{j'} \exp(z_{j'})$

Generalized Linear Models

(Inverse) Link function + likelihood. Many libraries handle them!

Regression	Inverse link function	Link function	Likelihood
Linear	identity	identity	Gaussian
Logistic	sigmoid	logit	Bernoulli
Poisson	exponential	log	Poisson
Negative binomial	exponential	log	Negative binomial

Recap

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  - Least squares = MLE
  - Ridge regression = MAP
- Overdispersion
  - Model mis-specification $\implies$ overly narrow uncertainty

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Next up: model checking and frequentist GLMs

Posterior Predictive Distribution

Have so far seen several Bayesian objects:

- Prior $p(\theta)$
- Posterior $p(\theta \mid x_{1:n})$

Another important object: **posterior predictive distribution** (PPD)

- Predict a new data point from data so far
- E.g.: $p(x_{n+1} \mid x_{1:n})$, or $p(y_{n+1} \mid x_{n+1}, y_{1:n}, x_{1:n})$

Trick for computing PPD

Suppose that $x_{n+1} \perp\!\!\!\perp x_1, \dots, x_{n+1} \mid \theta$ (Bayesian hierarchical model)

Then:

$$p(x_{n+1} \mid x_{1:n}) = \iint p(x_{n+1} \mid \theta) \underbrace{p(\theta \mid x_{1:n})}_{\text{posterior for } \theta} d\theta \tag{1}$$

$\implies$ Draw samples θ from posterior, then sample x_{n+1} conditioned on θ.

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More general idea: add x_{n+1} as a variable in PyMC3, and just sample it along with everything else

- Pro: works for any model structure
- Con: have to decide in advance which predictions you care about

Bayesian Model Checking via PPD

PPD gives us ways of **checking** a model.

Intuition: check that predicted data "looks like" real data

Examples:

- Check that predicted y_i look like true y_i in regression
- Check predictions on a hold-out set

Posterior Predictive Checks

Brainstorming exercise: What are ways we could formalize whether predicted samples "look like" real data?

(What statistics would you compute?)

Posterior Predictive Checks: Jupyter demo

[Demo with wind turbine data]

Frequentist GLMs

So far, looked at GLMs in Bayesian framework: prior + posterior + inference

Works equally well in frequentist world: just drop prior and use MLE!

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I.e. for Poisson:

$$\hat{\beta}_{\text{MLE}} = \operatorname{argmin}_{\beta} \frac{1}{n} \sum_{i=1}^{n} \log P_{\text{Poisson}}(y_i; \exp(\beta^\top x_i)) \tag{2}$$

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There is also a package for handling this: `statsmodels`.

Statsmodels Demo

[Jupyter wind turbine demo]

Model Checking for MLE

No prior, so no posterior. Are there other types of predictive checks we can use?

[Brainstorming exercise]

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Example: Chi-square statistic

$$\sum_{i=1}^{n} \frac{(y_i - \mathbb{E}[y \mid x_i, \hat{\beta}])^2}{\mathrm{Var}[y \mid x_i, \hat{\beta}]} \tag{3}$$

Frequentist Model Checking: Summary

- Log-likelihood: larger (less negative) for better models
- Deviance: for n datapoints and p parameters, should be roughly $n - p$ (assuming model is correct)
- Chi-square statistic: also should be roughly $n - p$. (Why not n?)

Overall Summary

Many tools for model checking:

- Bayesian: posterior predictive checks
- Frequentist: chi-square statistic

All models are "wrong" to some degree. These tools tell us if things are "obviously" wrong.