Lecture 24: Nonparametric Models

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Motivation

Recall linear regression / classification setup:

$$L(\beta) = \frac{1}{n}\sum_{i=1}^{n}(y^{(i)} - \beta^\top x^{(i)})^2 \text{ (linear)}$$

$$L(\beta) = \frac{1}{n}\sum_{i=1}^{n} -\log\sigma((-1)^{y^{(i)}}\beta^\top x^{(i)}) \text{ (logistic)}$$

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(E.g. true function not linear in x)

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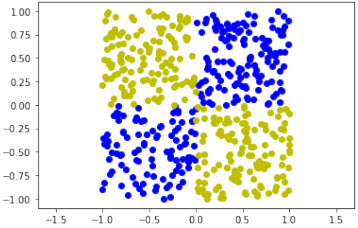
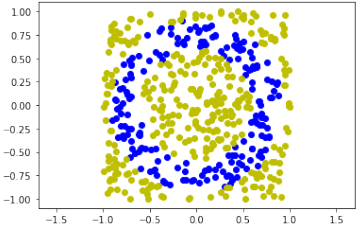
Non-linear Examples





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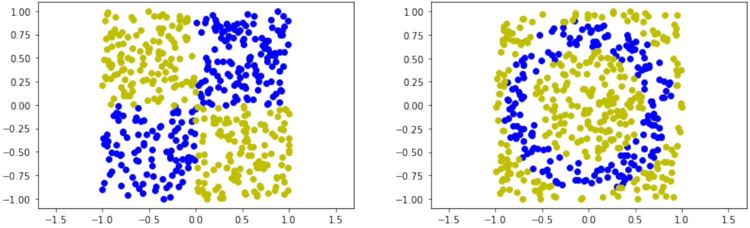

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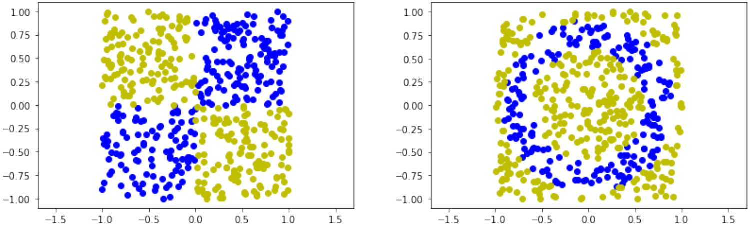

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- This gets tedious.
- What if we can't think of good features ahead of time?

Non-parametric modeling

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- Neural networks
- Kernels
- Decision trees

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Focus on first two for this lecture

Random features

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Solution: make ϕ random but high-dimensional:

$$\phi(x) = \operatorname{sign}(Mx + b), \tag{1}$$

where $M \in \mathbb{R}^{d \times k}$ and $b \in \mathbb{R}^k$ are random vectors (chosen once at beginning).

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Other features work too, e.g. $\cos(Mx + b)$, etc. Key points are randomness (good variation) and high dimensionality (usually $k > d$).

Random features: Jupyter demo

[switch to notebook]

Learned features

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Two-layer neural network:

$$\begin{aligned} \phi(x) &= \sigma(M_1 x + b_1), \\ p(y \mid x) &= \sigma(M_2 \sigma(M_1 x + b_1) + b_2). \end{aligned}$$

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Two-layer neural network:

$$\phi(x) = \sigma(M_1 x + b_1),$$
$$p(y \mid x) = \sigma(M_2 \sigma(M_1 x + b_1) + b_2).$$

Modern ML: iterate to many layers (and use different non-linearity σ, convolutional structure, etc.)

Learned features: Jupyter demo

[switch to notebook]

Fitting a neural network model

How do we actually fit M and b?

Recall stochastic gradient descent: update parameters $w = (M_1, M_2, b_1, b_2)$ by following gradient of the loss $\nabla L(w)$:

$$w' \leftarrow w - \eta \nabla L(w)$$

How do we compute $\nabla L(w)$?

Computing the gradient

[on board]

Backpropagation and autodiffentiation

- Given any "computation graph", we can write down derivatives recursively using the chain rule
- Then solve using dynamic programming!
- This is called backpropagation or autodifferentiation, key idea in Pytorch and other libraries
- Will build this up starting with simple examples

Backprop: simple example

[on board: $(a+b)c^2$ example]

Backprop: two-layer network

[on board]

Backprop in pytorch

[Jupyter demo]