

Examples on Nash Equilibrium

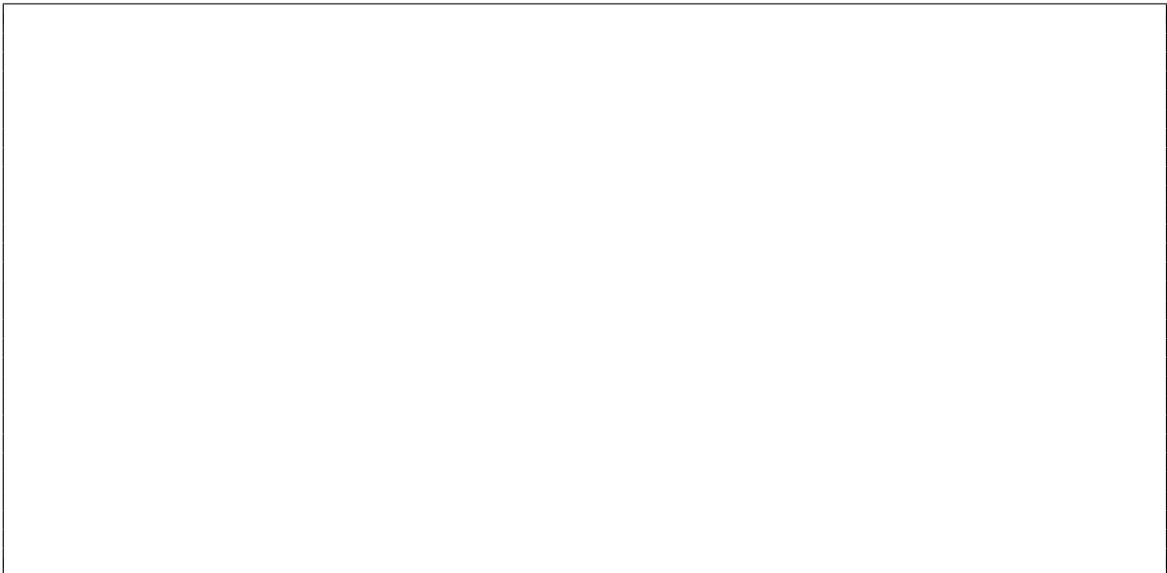
In class, we discussed the Nash equilibrium in two-player games. Denote the action space of player $i (i \in \{1, 2\})$ as \mathcal{A}_i . The payoff function (outcome) for player i is a function that maps the vector of actions taken by player 1, 2 to some real value $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \mapsto \mathbb{R}$. Each player i would like to maximize their own payoff function $u_i(a_1, a_2)$. We say the action pair (a_1^*, a_2^*) for the two players is a Nash equilibrium if $\forall a'_1 \in \mathcal{A}_1, u_1(a_1^*, a_2^*) \geq u_1(a'_1, a_2^*)$ and $\forall a'_2 \in \mathcal{A}_2, u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a'_2)$.

The definition of Nash Equilibrium can be extended to multi-player setting. Assume we have n players in total. Denote the payoff function for player i as $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n \mapsto \mathbb{R}$. Denote $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ as the vector of actions of all players except for player i . We say the action pair $(a_1^*, a_2^*, \dots, a_n^*)$ is a Nash equilibrium if for all $i \in [n], a'_i \in \mathcal{A}_i$, we have $u_i(a_i^*, a_{-i}^*) \geq u_i(a'_i, a_{-i}^*)$.

Consider the following games and provide one Nash equilibrium for each of them.

- (a) A two-player non-zero-sum game with payoff matrix as below.

		Player 1	
		0	1
Player 2	0	(3, 3)	(1, 0)
	1	(2, 2)	(5, 5)



- (b) A two-player zero-sum game. The action space for both players is $\mathcal{A}_i = \mathbb{R}$. We use $X, Y \in \mathbb{R}$ to denote the action of player 1 and 2, separately. The payoff function for player 1 is $u_1(X, Y) = Y^2 - X^2 + 2XY + 2X$. The payoff function for player 2 is $u_2(X, Y) = -u_1(X, Y) = -Y^2 + X^2 - 2XY - 2X$.



- (c) (Optional) A n -player single-item second-price auction. Denote the private valuation of the i -th bidder as $v_i \in \mathbb{R}^+$, the bid of the i -th bidder as $b_i \in \mathbb{R}^+$. The payoff function for bidder i is $u_i(b_1, b_2, \dots, b_n) = (v_i - \max_{j \neq i} b_j) \cdot 1(b_i \geq \max_{j \neq i} b_j)$. (Take some time to convince yourself that this payoff function is exactly the gain of bidder i from second-price auction.)

