DS 102 Discussion 2

Wednesdeay, September 9, 2020

1. **Bonferroni controls FWER.**

Suppose you have a test $T(X)$, where X is your data. Recall that if we observe a test statistic t, the p-value P of the test is:

$$P(X) = \mathbb{P}(T(X) > t)$$

, where $\mathbb{P}$ denotes the probability with respect to data drawn from the null distribution. This is the probability that, under the null distribution, you see a result at least as extreme as from your data. Remember that a p-value is a random variable, since it is a function of your data.

(a) Suppose you have two independent p-values p_1 and p_2. If on both hypotheses you choose a $0 < \alpha < 1$ and use the naive decision rule:

$$\delta(p; \alpha) = \begin{cases} \text{reject null} & p \leq \alpha \\ \text{accept null} & p > \alpha \end{cases}$$

what is the probability of making at least one false discovery? This probability is also known as the family-wise error rate (FWER).

Hint: recall that the distribution of p-value under null hypothesis is uniform.

Solution: A false discovery occurs if you reject the null hypothesis when the data comes from the null distribution. That is, $p_i < \alpha$ when the null is true. We have:

$$\mathbb{P}(\text{false discovery}) = \mathbb{P}(p \leq \alpha) = \alpha$$

If we have two p-values, the probability that we make a false discovery is:

$$\begin{aligned} \mathbb{P}(\text{at least one false discovery}) &= 1 - \mathbb{P}(\text{no false discovery}) \\ &= 1 - (1 - \alpha)^2 \\ &= \alpha(2 - \alpha) \end{aligned}$$

(b) Does this decision rule keep the probability of a false discovery below α?

Solution: No, since $\alpha(2 - \alpha) \geq \alpha$ for $0 < \alpha < 1$, we have that:

$$\mathbb{P}(\text{at least one false discovery}) > \alpha.$$

The naive decision rule doesn't even work with two p-values!

(c) The Bonferroni correction, which uses the decision rule

$$\delta\left(p; \frac{\alpha}{n}\right)$$

controls the FWER described in the previous problem. Suppose you have n independent p-values: $p_1, \dots, p_n$. Show that the Bonferroni correction controls the probability of at least one false discovery. *Hint: Let E_i be the event that $p_i < \frac{\alpha}{n}$. This is also shown in the lecture.*

Solution: Using the union bound,

$$\begin{aligned}\mathbb{P}(\text{at least one false discovery}) &= \mathbb{P}(\cup_{i=1}^{n} E_i)\\ &\leq \sum_{i=1}^{n} \mathbb{P}(E_i)\\ &\leq n\frac{\alpha}{n} = \alpha\end{aligned}$$

(d) Given 10 p-values for multiple hypotheses testing: 0.001, 0.003, 0.012, 0.015, 0.08,0.09, 0.1, 0.14, 0.16, 0.28. What threshold should be used for the decision rule such that the FWER is less than 0.05? How many tests are rejected?

Solution: The desired threshold is

$$\frac{\alpha}{n} = \frac{0.05}{10} = 0.005.$$

Any p-values that are smaller than this will be rejected, which is 0.001, 0.003. So 2 tests will be rejected.

2. **Benjamini-Hochberg procedure** In this question we analyze the properties of the Benjamini-Hochberg (BH) procedure. Recall the steps of the procedure:

Algorithm 1 The Benjamini-Hochberg Procedure

input: FDR level α, set of n p-values $P_1, \dots, P_n$

Sort the p-values $P_1, \dots, P_n$ in non-decreasing order $P_{(1)} \leq P_{(2)} \leq \cdots \leq P_{(n)}$

Find $K = \max\{i \in \{1, \dots, n\} : P_{(i)} \leq \frac{\alpha}{n} i\}$

Reject the null hypotheses (declare discoveries) corresponding to $P_{(1)}, \dots, P_{(K)}$

(a) Given 10 p-values for multiple hypotheses testing: 0.001, 0.003, 0.012, 0.015, 0.08,0.09, 0.1, 0.14, 0.16, 0.28. Suppose we would like to control the FDR at the level 0.05. How many tests are rejected?

Solution: By first sorting the p-values and compare the k-th p-value with $k\alpha/n = 0.005k$, we will see that the largest k such that its p-value is smaller than $0.005k$ is $k = 4$. So 4 tests will be rejected.

(b) Suppose $P_1 = P_2 = \cdots = P_n = \alpha$, and we run BH under level α on these p-values. How many discoveries does BH make? Explain.

Solution: It makes n discoveries, because the highest p-value (equal to α) is less than or equal to $\frac{\alpha}{n} n = \alpha$.

(c) Suppose $P_1 = P_2 = \cdots = P_{n-1} = \alpha, P_n = \alpha + 0.001\alpha$, and we run BH under level α on these p-values. How many discoveries does BH make? Explain.

Solution: It makes 0 discoveries, because no p-value is under the corresponding threshold $\frac{\alpha}{n}k$.

(d) Suppose we run BH on $\{P_1, \ldots, P_n\}$, and we make $R < n$ discoveries. Now suppose we add an extra p-value equal to 0 to this set. Now we run BH on $\{P_1, \ldots, P_n, 0\}$ and get a new number of rejections R'. Which of the following are possible: $R' > R$, $R' = R$, $R' < R$? If multiple are possible, list all that are possible. Explain why.

Solution: The new p-value 0 is now the smallest one in the sequence. Therefore, if some given p-value was compared to $\frac{\alpha}{n}k$ before adding the extra p-value, now it's compared to $\frac{\alpha}{n+1}(k+1)$. Since $\frac{\alpha}{n+1}(k+1) > \frac{\alpha}{n}k$, it is now strictly easier to discover. Moreover, the new p-value 0 will definitely be discovered, so $R' > R$ is the only possiblity.